SB-3532

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M. Sc. - II Examination

March / April – 2011

Mathematics

(Advanced Linear Algebra)

Time : 3 Hours] [Total Marks : 70

Instruction : (1)

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Name of the Examination :
☛ M. Sc. - 2

Name of the Subject :
☛ Mathematics

☛ Subject Code No. : 3 5 3 2 ☛ Section No. (1, 2,.....) : Nil

Seat No. :

Student's Signature

(2) This question paper contains five questions with options.
(3) Follow usual notations and conventions.

1 (a) Let $T:V \to W$ be a linear operator. Define the following terms : **6**

(i) Dimension of a vector space
(ii) Range of operator T;
(iii) Nullity of operator T.

The linear operator $T:V \to W$ is defined on a finite dimensional vector space V. Show that

$$\dim(R(T)) + \dim(N(T)) = \dim(V)$$

(b) Attempt any **two** out of three : **8**

(i) Define null space of a linear operator. Let V and W be vector spaces. Show that the null space and range of a linear operator $T:V \to W$ are subspaces of V and W, respectively.

(ii) Given $P_n \equiv span\left\{1, t, t^2, \ldots, t^n\right\}$ is the set of polynomials of degree not exceeding n. Let the operator $T:P_n \to R^{n+1}$ be defined by

$T(f) = x = (x_0, x_1, \ldots, x_n)$, where

$x_k = \int_0^1 t^k (1-t)^{n-k} f(t)\,dt$; $k = 0, 1, 2, \ldots, n$.

Show that T is invertible.

(iii) Let $V = P_2$ and $T : V \to V$ be defined by

$$(Tp)(x) = \frac{1}{2}\int_{-1}^{1}\left(15t^2 + 3tx + 6x^2\right)p(t)\,dt.$$

Using the standard basis $\beta = \left\{1, t, t^2\right\}$, construct a matrix $[T]$ of operator T. Verify that

$$[T]^{-1} = \begin{bmatrix} \frac{-1}{4} & 0 & \frac{3}{8} \\ 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{-5}{8} \end{bmatrix},$$ and compute $T^{-1}(1)$, $T^{-1}(t)$

and $T^{-1}\left(t^2\right)$.

2 (a) Given $\left\{v_1, v_2, \ldots, v_n\right\}$, a set of linearly independent vectors in V equipped with the inner product $\langle\ ,\ \rangle$. The Gram-Schmidt process of orthogonalisation produces an orthogonal set $\left\{e_1, e_2, \ldots, e_n\right\}$ using the steps described below : **6**

Step 0

Set $e_1 = v_1$ and compute $\left\| e_1 \right\|^2$.

Step k

* Suppose $e_1, e_2, \ldots, e_k$ have been constructed and $\left\| e_i \right\|^2$, $i = 1, 2, \ldots, k$ have been computed.

* Compute the Fourier sum of v_{k+1} with respect to $\left\{e_1, e_2, \ldots, e_n\right\}$: $s_{k+1} = \sum_{j=1}^{k} d_j e_j$ where

$$d_j = \frac{\left\langle v_{k+1}, e_j \right\rangle}{\left\| e_j \right\|^2}; \; j = 1, 2, \ldots, k.$$

* Set $e_{k+1} = v_{k+1} - s_{k+1}$ and compute $\| e_{k+1} \|^2$ directly

or using $\| e_{k+1} \|^2 = \| v_{k+1} \|^2 - \sum_{j=1}^{k} |d_j|^2 \| e_j \|^2$.

Let $\{v_0, v_1, v_2, v_3\}$ be a set of linearly independent vectors defined by $v_i(t) = t^i ; i = 0, 1, 2, 3$ and the inner product be defined by $\langle x(t), y(t) \rangle = \int_0^{\infty} x(t)\overline{y(t)} e^{-t} dt$.

Construct a set $\{e_0, e_1, e_2, e_3\}$ of orthogonal polynomials, using the Gram-Schmidt process outlined above.

(b) Attempt any **2** out of 3 : **8**

(i) Define the following terms :

(1) Inner product

(2) Norm generated by an inner product.

Show that for a vector space V, $|\langle x, y \rangle| \le \| x \| \| y \|$, for all $x, y \in V$.

(ii) Explain what is meant by QR factorization/ decomposition of matrix A. Find the QR factorization of the matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$.

(iii) Find a linear combination s of vectors $v_1 = (1, 0, -2, 1)^T$, $v_2 = (2, 1, -2, 0)^T$ and $v_3 = (1, -1, -1, 3)^T$, for which $\| f - s \|$ is minimal, where $f = (2, 3, 2, 2)^T$. Use standard inner product for the calculations.

3 (a) Let $M = span\{A_1, A_2, \ldots, A_n\}$, where $A_i \in F^m$, $i = 1, 2, \ldots, n$, and let y be given. Further assume that $A = [A_1 A_2 \ldots A_n]$ and consider the standard inner product $\langle\ ,\ \rangle$ on F^m. Show that the following statements are equivalent : **6**

(i) $s = Ax$ is the projection of y on M

(ii) $z = x$ minimizes $\| y - Az \|$, where $z \in F^n$.

(iii) x satisfies the normal equations $A^H Ax = A^H y$

(b) Attempt any **2** out of 3 : **8**

(i) Let G be the Gram matrix of $\{v_1, v_2, \ldots, v_n\}$. Show that $r(G) = \dim\left(span\left(\{v_1, v_2, \ldots, v_n\}\right)\right)$

(ii) Let A be an $m \times n$ matrix. Given the standard inner product $\langle\ ,\ \rangle$ on F^n, show that

(1) $N(A) = R(A^H)^{\perp}$

(2) $R(A) = N(A^H)^{\perp}$

(3) $r(A^H) = r(A)$

(iii) Define the orthogonal complement of an arbitrary set M. Given subspace

$M = span\{(1, 1, 1, 1)^T, (1, 2, 2, -1)^T, (1, 0, 0, 3)^T\}$,

express $x = (4, -1, -3, 4)^T$ as $x = s + r$, where $s \in M$ and $r \in M^{\perp}$.

4 (a) Explain what is meant by algebraic and geometric multiplicities of an eigen value. Let $V = P_2$ and $T : V \to V$ be defined by **6**

$$(Tp)(x) = \frac{3}{8}\int_{-1}^{1}\left[1+4(x+t)+5\left(x^2+t^2\right)-15x^2t^2\right]p(t)\,dt$$

Find an eigenbasis for T and hence show that T is diagonalizable.

(b) Attempt any **2** out of 3 : **8**

(i) Let the operator $T : V \to V$ be defined by $T(f) = f''$ where $' = \frac{d}{dx}$. Show that T is diagonalizable on $V = T_n$ but not on $V = P_n, (n > 1)$.

(ii) The rotation matrix R is defined by $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ whre θ is the angle of rotation. Show that the matrix R will have complex eigenvalues if θ is not a multiple of π. Also find the spectrum and the eigenspaces of R.

(iii) The Gershgorin's theorems for estimating the range of eigenvalues are stated below :

Result 1 :

For an eigenvalue λ of matrix

$$A = \left[a_{ij}\right], |\lambda| \le \max\left\{\sum_{i=1}^{n}\left|a_{ij}\right| : j = 1, 2, \ldots, n\right\}$$

Result 2 :

For an $n \times n$ matrix $A = \left[a_{ij}\right]$, if the row radii R_i is defined by $R_i = \sum_{j=1,\, j\ne i}^{n}\left|a_{ij}\right|, i = 1, 2, \ldots, n$ and the corresponding closed disk in the complex plane as

$D_i = \{z : |z - a_{ii}| \le R_i\}$, then each eigenvalue λ of A lies in the union of these disks.

(1) Apply the above results to estimate, and show in figure, the bounds for eigenvalues of the matrix $A = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 4 & 4 \\ 3 & 4 & 6 \end{bmatrix}$.

(2) Estimate the largest and the smallest eigenvalues of A using Rayleigh Quotient method, first by considering $x = (1, 1, 1)^T$ and then $x = (-1, 2, -1)^T$ as your guesses.

5 (a) Let V be a vector space with $\dim(V) < \infty$. Let $T : V \to V$ be a linear operator, normal with respect to the inner product $\langle \, , \rangle$ on V. Let $\sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ be the distinct eigenvalues of T and $M = M(\lambda_i)$ the corresponding eigenspaces. Show that every $x \in V$ can be written uniquely as $x = \sum_{i=1}^{r} v_i$ for some $v_i \in M_i$. **6**

Further, for each i, v_i is the orthogonal projection of x on M_i.

(b) Attempt any **2** out of 3 : **8**

(i) Define : Unitary operator.

Let $T : V \to V$ be a linear operator defined on a vector space V with inner product $\langle \, , \rangle$ and $\dim(V) < \infty$. Show that the following statements are equivalent.

(i) T in unitary on V with respect to $\langle \, , \rangle$

(ii) $T*T = I$

(iii) $(Tx, Ty) = \langle x, y \rangle$ for all $x, y \in V$

Show that $T : T_n \to T_n$, define by $Tf(x) = f(x + \alpha)$, where $\alpha \in R$ and the inner product defined by $\langle f, g \rangle = \int_0^{2\pi} f(x)\, \overline{g(x)}\, dx$, is unitary.

(ii) Define the Rayleigh quotient of an operator T with respect to the inner product $\langle \, , \rangle$. Let T be Hermitian on V and let $\{e_k\}_{k=1}^{n}$ be orthogonal with respect to $\langle \, , \rangle$, $T(e_i) = \lambda_i e_i$ and $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$. If $0 \ne x = \sum_{i=1}^{n} x_i e_i$, show that $\lambda_1 \le R(x) \le \lambda_n$.

(iii) Show that for eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ of an $n \times n$ matrix A.

(1) $\prod_{i=1}^{n} \lambda_i = \det(A)$

(2) $\sum_{i=1}^{n} \lambda_i = tr(A)$

Use the above results, to show that for a 2×2 matrix A, if $p(\lambda) = \lambda^2 + b\lambda + c$ is the characteristic polynomial of A, then $b = -tr(A)$ and $c = \det(A)$.